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Journal of Sound and Vibration 286 (2005) 663–668
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

A conserved quantity and the stability of axially moving nonlinear beams

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Received 16 September 2004; received in revised form 24 December 2004; accepted 10 January 2005
Available online 21 March 2005

Abstract

Free nonlinear transverse vibration is investigated for an axially moving beam modeled by an integro-partial-differential equation. Based on the equation, a conserved quantity is defined and confirmed for axially moving beams with pinned or clamped ends. The conserved quantity is applied to demonstrate the Lyapunov stability of the straight equilibrium configuration in transverse nonlinear of beam with a low axial speed.

1. Introduction

Axially moving beams can represent many engineering devices [1,2]. Despite many advantages of these devices, noise and vibration, particularly transverse vibration, associated with the devices have limited their applications. For example, in belt drive systems, the vibration of the belt leads to noise and accelerated wear of the belt; in band saws, the vibration of the blade results in poor cutting quality. Therefore, understanding transverse vibrations of axially moving beams is important for the design of the devices.

The transverse motion of an axially moving beam can be regarded as free vibration if both external excitations and parametric excitation are not taken into consideration. It is well known

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.01.011

that the total mechanical energy in free vibration of an undamped nontranslating beam with pinned or fixed ends is constant. However, many researchers found that the total mechanical energy associated with free vibration of an axially moving beam is not constant when the beam travels between two motionless supports. Barakat [3] considered the energetics of an axially moving beam and found that energy flux through the supports can invalidate the linear theories of axially moving beams at sufficiently high transporting speed. Tabarrok et al. [4] showed that the total energy of a traveling beam without tension is periodic in time. Wickert and Mote [5] presented the temporal variation of the total energy related to the local rate of change and calculated the temporal variation of energy associated with modes of moving beams. Considering the case that there were nonconservative forces acting on two boundaries, Lee and Mote [6] presented a generalized treatment of energetics of translating beams. Renshaw et al. [7] examined the energy of axially moving beams from both Lagrangian and Eulerian views and found that Lagrangian and Eulerian energy functionals are not conserved for axially moving beams. Zhu and Ni [8] investigated energetics of axially moving strings and beams with arbitrarily varying lengths. Hence the variation of the total mechanical energy is a fundamental feature of free transverse vibration of axially moving beams.

Although the total mechanical energy of axially moving materials is generally not constant, there does exist an alternative conserved quantity. Renshaw et al. [7] presented both Eulerian and Lagrangian conserved functionals for axially moving beams. Chen and Zu [9] generalized their results to nonlinear free vibration of axially moving beams. They adopted the simplest model, a partial-differential equation developed by Thurman and Mote [10] for axially moving materials undergoing nonlinear transverse vibration. The models are based on the assumptions that the plane transverse is not coupled with the longitudinal motion, and the vibration amplitude is small so that only the lowest order nonlinear term is retained in the governing equation. However, the widely used model for free vibration of axially moving beam is an integro-partial-differential equation derived from uncoupling the governing equation for coupled longitudinal and transverse vibration by neglecting the fast dynamics in the longitudinal direction [11–15]. Such a nonlinear model was also used in analysis of parametric vibration [16] and forced vibration [17]. In the present investigation, it will be demonstrated that there exists a conserved quantity in the free nonlinear transverse vibration governed by the integro-partial-differential equation. The conserved quantity will be applied to verify that the straight equilibrium configuration of an axially moving beam is stable in the Lyapunov sense for low axial speed.

2. A conserved quantity in vibration of an axially moving beam

Consider a uniform axially moving beam of linear density ρ, cross-sectional area A, cross-sectional area moment of inertial I, Young's modulus E, and initial tension P. The beam travels at the constant and uniform axial transport speed V between two boundaries separated by distance L. The distance from the left boundary is measured by fixed axial coordinate X, and time is denoted by T. The transverse displacement of the beam is given by the Eulerian variable $U(X, T)$ in the sense that $U(X, T)$ describes the displacement of the beam element instantaneously located at X even though different material elements occupy that position at different times. The governing equation in the dimensionless form is [11–15]

$$\frac{\partial^2 u}{\partial t^2} + 2v\,\frac{\partial^2 u}{\partial x \partial t} - (1 - v^2)\,\frac{\partial^2 u}{\partial x^2} + v_f^2\,\frac{\partial^4 u}{\partial x^4} = \frac{v_1^2}{2}\,\frac{\partial^2 u}{\partial x^2}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x, \tag{1}$$

where

$$u = \frac{U}{L}, \quad x = \frac{X}{L}, \quad t = \frac{T}{l}\sqrt{\frac{P}{\rho}}, \quad v = V\sqrt{\frac{\rho}{P}}, \quad v_f = \frac{1}{l}\sqrt{\frac{\mathrm{EI}}{P}}, \quad v_1 = \sqrt{\frac{\mathrm{EA}}{P}}. \tag{2}$$

Define a functional in a specified spatial domain (0, 1):

$$I = \int_0^1 \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}(1 - v^2)\left(\frac{\partial u}{\partial x}\right)^2 + \frac{v_f^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{v_1^2}{8}\left(\frac{\partial u}{\partial x}\right)^2 \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\right] \mathrm{d}x. \tag{3}$$

The time rate of change of the functional is

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}(1 - v^2)\left(\frac{\partial u}{\partial x}\right)^2 + \frac{v_f^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{v_1^2}{8}\left(\frac{\partial u}{\partial x}\right)^2 \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\right] \mathrm{d}x. \tag{4}$$

Here, the order of differentiation and integration can be interchanged, as the limit of integration is time independent. After some mathematical manipulations, one can cast Eq. (4) into the form

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} = {} & \int_0^1 \frac{\partial u}{\partial t}\left[\frac{\partial^2 u}{\partial t^2} + 2v\,\frac{\partial^2 u}{\partial x \partial t} - (1 - v^2)\,\frac{\partial^2 u}{\partial x^2} + v_f^2\,\frac{\partial^4 u}{\partial x^4} - \frac{v_1^2}{2}\,\frac{\partial^2 u}{\partial x^2}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\right] \mathrm{d}x \\ & + \left[\frac{\partial u}{\partial t}\left(-v\,\frac{\partial u}{\partial t} - v_1^2\,\frac{\partial^3 u}{\partial x^3} + \frac{v_1^2}{2}\,u_{,x}\int_0^L u_{,x}^2\,\mathrm{d}x\right) + v_1^2\,\frac{\partial^2 u}{\partial x^2}\,\frac{\partial}{\partial t}\,\frac{\partial u}{\partial x}\right]\Bigg|_0^1. \end{aligned} \tag{5}$$

From the governing equation (1), one gets the time rate of change of the functional (4)

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \left[\frac{\partial u}{\partial t}\left(-v\,\frac{\partial u}{\partial t} - v_f^2\,\frac{\partial^3 u}{\partial x^3} + \frac{v_1^2}{2}\,u_{,x}\int_0^L u_{,x}^2\,\mathrm{d}x\right) + v_f^2\,\frac{\partial^2 u}{\partial x^2}\,\frac{\partial}{\partial t}\,\frac{\partial u}{\partial x}\right]\Bigg|_0^1. \tag{6}$$

If the beam is constrained by simple or fixed supports, the boundary conditions in the dimensionless form are

$$u(0, t) = u(1, t) = 0, \quad \frac{\partial u}{\partial x} = 0 \text{ or } \frac{\partial^2 u}{\partial x^2} = 0 \quad (x = 0, 1). \tag{7}$$

In both cases, Eq. (6) leads to

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0. \tag{8}$$

Thus, I remains a constant during the free transverse vibration of an axially moving beam governed by Eq. (1).

If $v_1 = 0$, Eq. (3) gives the corresponding result in linear transverse vibration of an axially moving beam obtained by Renshaw et al. [7]. If $v_f = 0$, Eq. (3) yields the corresponding result in transverse vibration of an axially moving Kirchhoff string [18].

Here, the quantity I is defined from the Eulerian view, which is concerned with a specified spatial domain. A Lagrangian conserved functional, which is concerned with a specified set of particles, can be similarly defined. As Renshaw et al. [7] concluded, a conserved Eulerian functional qualifies as Lyapunov functionals in stability analysis, while a conserved Lagrangian functional cannot serve as Lyapunov functionals because their time derivatives are only defined at an instant. Therefore, this Letter discusses only the conserved Eulerian functional.

3. The stability of the straight equilibrium position

The conserved quantity (3) can be used to demonstrate the stability of the straight equilibrium configuration of a beam moving with low axial speed. That is, the resulting vibration about the equilibrium produced by a small initial disturbance will be small.

Define a norm for a function f on $[0, L]$:

$$||f|| = \int_0^1 \left[f^2(x) + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)^2 \right] \mathrm{d}x. \tag{9}$$

For the solution $u(x, t)$ of Eq. (1) under the boundary condition (7) and the initial conditions

$$u(x,0) = \alpha(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \beta(x) \tag{10}$$

if the axial speed is satisfied with $v<1$, then there exists a positive number k such that

$$||u(x,t)|| \leqslant k[||\alpha(x)|| + ||\beta(x)|| + ||\alpha(x)||^2]. \tag{11}$$

In fact, the Hölder inequality and the boundary conditions (7) yield

$$\int_0^1 u^2(t,x)\,\mathrm{d}x = \int_0^1 \left[\int_0^x \frac{\partial u}{\partial x}\,\mathrm{d}x\right]^2 \mathrm{d}x \leqslant \int_0^1 \left[\sqrt{x}\int_0^x \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\right] \mathrm{d}x = \frac{2}{3}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x. \tag{12}$$

Notice that under the condition $1 - v^2 > 0$. Eqs. (9), (12) and (3) lead to

$$\begin{aligned} ||u(x,t)||^2 &= \int_0^1 \left[u^2(x,t) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \right] \mathrm{d}x \leqslant \int_0^1 \left[\frac{5}{3}\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \right] \mathrm{d}x \\ &\leqslant k_1 \int_0^1 \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}(1-v^2)\left(\frac{\partial u}{\partial x}\right)^2 + \frac{v_f^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{v_1^2}{8}\left(\frac{\partial u}{\partial x}\right)^2 \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \right] \mathrm{d}x \\ &= k_1 I, \end{aligned} \tag{13}$$

where

$$k_1 = \max\left\{ \frac{10}{3(1-v^2)}, \frac{2}{v_f^2} \right\}. \tag{14}$$

As I is conserved during the vibration, its value remains a constant that can be calculated at the initial time. Under the initial condition (10),

$$I = \int_0^1 \left[\frac{1}{2}\beta^2(x) + \frac{1}{2}(1-v^2)\left(\frac{\mathrm{d}a}{\mathrm{d}x}\right)^2 + \frac{v_f^2}{2}\left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}\right)^2 + \frac{v_1^2}{8}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2 \int_0^1 \left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2 \mathrm{d}x\right] \mathrm{d}x. \tag{15}$$

Based on the definition (9)

$$I \leqslant k_2[||\alpha(x)|| + ||\beta(x)|| + ||\alpha(x)||^2], \tag{16}$$

where

$$k_2 = \max\left\{\frac{1}{2}, \frac{v_f^2}{2}, \frac{v_1^2}{2}\right\}. \tag{17}$$

Let

$$k = k_1 k_2. \tag{18}$$

Then, inequality (11) holds as the result of inequalities (13) and (16).

Inequality (11) means that the resulting vibration of small initial disturbance will be small. Hence, the straight equilibrium configuration of an axially moving beam is stable in the Lyapunov sense.

4. Conclusions

This Letter investigates free nonlinear transverse vibration of an axially moving beam modeled by an integro-partial-differential equation. Based on the equation, a conserved quantity is defined and confirmed for axially moving beams with pinned or clamped ends. The conserved quantity defined here reduces to the corresponding known result of linear vibration of an axially moving beam if the nonlinear term is dropped out. The conserved quantity is applied to verify the Lyapunov stability of the straight equilibrium configuration in transverse nonlinear vibration of an axially moving beam.

Acknowledgment

The research is supported by the National Natural Science Foundation of China (Project No. 10472060) and Natural Science Foundation of Shanghai Municipality (Project No. 04ZR14058).

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